DIFFUSION APPROXIMATION IN CALCULATIONS OF THE TRANSFER OF RADIANT ENERGY

S. P. Detkov

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The methodological aspect of diffusion approximation is analyzed. The effect of the function characterizing the radiation of the medium, of the boundary conditions, of the optical depths, and of the thicknesses on approach to the plane layer on the coefficient of diffusion is investigated. Recommendations are given with respect to the application of the diffusion approximation.

The analogy between photon propagation and the diffusion of molecules in an admixture in a gas was apparently first validated in [1, 2] for reemission and in [3] for scattering. The analogy was then extended to neutron flows [4–7]. The diffusion approximation is treated in [8] as a consequence of the phenomenological differential equation of the transfer of radiant energy; in [9] it is treated as a consequence of the "forward-backward" approximation for average cosines equal to $1/(3)^{1/2}$; in [6, 7, 10, 11] it is treated as the first term of the expansion in the method of spherical harmonics; in [12] it develops that the diffusion approximation results from consideration only of the first derivative in the expansion in Taylor series of the sought function in the integral equation; it is presented in [13] in the asymptotic form of the integral equation of transfer for points distant from the boundaries. Here the diffusion approximation will be treated as an independent phenomenological equation of transfer, in which the density of the radiant energy is taken as the potential of the transfer field. In addition to simplicity, this results in freedom in expressing the diffusion coefficient

$$\mathbf{q} = -\frac{c}{mk}\,\mathrm{grad}\,U. \tag{1}$$

From the mechanism of diffusion in an infinite medium for $\mathrm{div}\,\mathbf{q} = 0$ we have m = 3. The number m may vary significantly at the boundaries at distances <2–3 mean free paths. In the case of the "forward-backward" approximation with the average cosine 0.5, m = 4 [14, 15]. In [16, 17], etc., m = 3 is recommended for the core of the flow and m = 4 is recommended for the points close to the boundaries. However, the value of m frequently exceeds the limits of this interval. A significant refinement of the calculations results from variation of the boundary conditions. It is recommended in [6, 18, 19] that a factor determined by the solution of a simple problem be added to the magnitude of the radiant-energy density at the boundary; it is demonstrated that this may result in excellent agreement with exact solutions. The most exact boundary conditions are determined iteratively [20, 21]. However, in addition to the substantial increase in complexity, we note that this refinement affects the heat flow exclusively. The effect for the temperature field is insignificant. The 3% accuracy for the temperature field in [20] is apparently accidental, since the T^4 field is linear with the same accuracy. In [22] (unlike the analogous problem in [23]) the exact boundary conditions are used because of symmetry, but the error in the determination of the temperatures was great. The most exact boundary conditions are evidently not sufficient to offset the fact of the variation in the number m with the coordinate [22]. A physical foundation is provided in [24] for the theorem on the change in the number m with the coordinate and the function $m(\tau, \tau_0)$ is studied for two of the simplest problems. The basis for the criticisms of the diffusion approximation in [22, 25, 26, etc.] should be sought not only in the boundary conditions, but also in the changes in the number m.

Limiting ourselves to one-dimensional problems, on the basis of the exact differential equation [27–29]

$$\mathbf{q} = -\frac{c}{k}\,\mathrm{div}\,\mathbf{P} \tag{2}$$

we obtain

$$\mathbf{q} = -\frac{c}{k}\,\mathrm{grad}\,P_{11}. \tag{3}$$

Thus in actual fact the transfer potential is a component of the radiant-pressure tensor

$$P_{11} = \frac{1}{c}\int\limits_{4\pi} I\cos^2\theta\, d\omega. \tag{4}$$

Unlike the exact equation (3), Eq. (1) exhibits an advantage in the direct relationship between the transfer "potential" and temperature:

$$cU = 4\sigma T^4 - \frac{g_0}{\alpha}, \tag{5}$$

thanks to which the diffusion approximation remained viable, despite all of its shortcomings. It is recommended in [20, 21] that Eq. (2) or (3) with exact condition at the boundary serve as the basis. Here the temperature field, without which the application of (2) or (3) is impossible, is determined from (1) and (5). As a result we see that: a) without the diffusion approximation the solution of (2) and (3) is possible only through the trial-and-error method or by resorting to an integral equation and, thus, (2) or (3) cannot be the equivalent of an integral equation; b) for the actual utilization in [20, 21] of the diffusion approximation, everything is additionally reduced to the selection of the boundary conditions. We return to the inadequacies of the boundary conditions in the light of the variable number m.

Here the function $m(\tau, \tau_0)$ is studied in approximation of a plane layer in order to develop additional recommendations. The first method for the analytical study of the function $m(\tau)$ involves replacement of the boundary surfaces by an equivalent radiating medium which subsequently extends through the interval $[-\infty, \infty]$. If the function $B_{ef}(\tau)$ in this interval has a sufficient number of derivatives, it is possible to expand it in Fourier series with respect to the point τ and additionally to present in power series

$$B_+(\tau') = a + f_1(u) + f_2(u),$$

$$B_-(\tau') = a + f_1(u) - f_2(u),$$

where $f_1(u)$ and $f_2(u)$ are, respectively, the even and odd power functions corresponding to the cosine and sine series. Bearing in mind that

$$I_+(\tau, \mu) = \int_\tau^\infty B_+(\tau') \exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu},$$

$$I_-(\tau, \mu) = \int_{-\infty}^\tau B_-(\tau') \exp\left(-\frac{\tau-\tau'}{\mu}\right)\frac{d\tau'}{\mu},$$

$$cP_{11} = 2\pi \int_0^1 (I_+ + I_-)\mu^2 d\mu,$$

$$cU = 2\pi \int_0^1 (I_+ + I_-) d\mu,$$

we obtain

$$m = \int_0^\infty f_2(u)\exp(-u)\frac{du}{u}\left[\int_0^\infty f_2(u)E_2(u)\,du\right]^{-1}.$$

As we can see, the number m is determined exclusively by the odd portion of the function $B_{ef}(u)$. Assuming $f_2(u) = \sum_{k=0}^{\infty} b_k u^{2k+1}$, we have

$$m = \sum_{k=0}^{\infty}(2k)!\,b_k\left(\sum_{k=0}^{\infty}\frac{(2k+1)!\,b_k}{2k+3}\right)^{-1}.$$

If the function $f_2(u)$ is linear, then m = 3. Generally, however, the coefficients b_k are not easily determined. Moreover, the function $B_{ef}(u)$ is not always smooth, particularly near the surfaces. The second method of investigation is therefore more convenient, involving the direct consideration of the boundary conditions.

The potentials of transfer in (1) and (3) have the form:

$$cU = 2q_{ef\,1}E_2(\tau) + 2q_{ef\,2}E_2(\tau_0-\tau) + 2\pi\int_0^{\tau_0} B_{ef}(\tau')E_1|\tau'-\tau|\,d\tau', \tag{6}$$

$$cP_{11} = 2q_{ef\,1}E_4(\tau) + 2q_{ef\,2}E_4(\tau_0-\tau) + 2\pi\int_0^{\tau_0} B_{ef}(\tau')E_3|\tau'-\tau|\,d\tau'. \tag{7}$$

Joint solution of (1), (3) and (6), (7) yields

$$m(\tau, \tau_0) = \left\{E_1(\tau) - q_2E_1(\tau_0-\tau) + \int_0^\tau B_*(\tau') \times \exp[-(\tau-\tau')]\frac{d\tau'}{\tau-\tau'} - \int_\tau^{\tau_0} B_*(\tau')\times \exp[-(\tau'-\tau)]\frac{d\tau'}{\tau'-\tau}\right\}\left\{E_3(\tau) - q_2E_3(\tau_0-\tau) + \int_0^\tau B_*(\tau')E_2(\tau-\tau')\,d\tau' - \int_\tau^{\tau_0} B_*(\tau')E_2(\tau'-\tau)\,d\tau'\right\}^{-1}.$$

The presence in the numerator of the function $E_1(u)$ for which, as $u \to 0$, $E_1(u) \to \infty$ results in singularities in the function $m(\tau, \tau_0)$. These arise at the points with temperature discontinuities as, for example, at the boundaries in the case of a medium of low thermal conductivity or at points with discontinuities in the function B_*. These singularities are localized by small values of u, since only for small u does $E_1(u)$ assume large values. When B_* = const

$$m(\tau, \tau_0) = \frac{E_1(\tau)(1-B_*) + E_1(\tau_0-\tau)(B_*-q_2)}{E_3(\tau)(1-B_*) + E_3(\tau_0-\tau)(B_*-q_2)}.$$

Here, for the middle of the layer

$$m\left(\frac{\tau_0}{2}\right) = \frac{E_1(\tau_0/2)}{E_3(\tau_0/2)} \tag{8}$$

and, as we can see, for the points most removed from the boundaries, a change in m is possible over a wide interval of values; the number m is independent here of B_* and q_2.

For the plane layer when div q = 0 the function $B_*(\tau)$ is linear, with an error of 3% [30, 31]:

$$\frac{B_*-q_2}{1-q_2} = a_* - b_*\tau,$$

where

$$a_* = \frac{1-\exp(-\tau_0) + 2\tau_0 - \tau_0E_2(\tau_0)}{2[1-\exp(-\tau_0)+\tau_0]},$$

$$b_* = \frac{1-E_2(\tau_0)}{1-\exp(-\tau_0)+\tau_0}.$$

Then

$$m(\tau, \tau_0) = \{E_1(\tau)(1-a_*+b_*\tau) + E_1(\tau_0-\tau)\times(a_*-b_*\tau) + b_*[2-\exp(-\tau) - \exp(-\langle\tau_0-\tau\rangle)]\}\left\{E_3(\tau)(1-a_*+b_*\tau) + E_3(\tau_0-\tau)(a_*-b_*\tau) + b_*\left[\frac{2}{3} - E_4(\tau) - E_4(\tau_0-\tau) - \tau E_3(\tau) - (\tau_0-\tau)E_3(\tau_0-\tau)\right]\right\}^{-1}.$$

For the middle of the layer

$$m\left(\frac{\tau_0}{2}\right) = \left\{E_1\left(\frac{\tau_0}{2}\right)[1+\tau_0-\exp(-\tau_0)] + \right.$$

$$+2\left[1-E_2(\tau_0)\right]\left[1-\exp\left(-\frac{\tau_0}{2}\right)\right]\Big\}\Big\{E_3\left(\frac{\tau_0}{2}\right)\times$$

$$\times\left[1+\tau_0-\exp(-\tau_0)\right]+\frac{2}{3}\left[1-E_2(\tau_0)\right]\times$$

$$\times\left[1-\exp\left(-\frac{\tau_0}{2}\right)-\tau_0 E_3\left(\frac{\tau_0}{2}\right)\right]\Big\}^{-1}.$$

Here the number m changes in the interval $[\infty, 3]$ and unlike the case with $B_* = \text{const}$ on the segment $[0, \tau_0]$ there are no values less than 3. For $\tau_0 \gg 1$ and points not close to the boundary, $m \simeq 3$. For $\tau_0 \ll 1$ and $\tau = \tau_0/2$ we obtain the earlier relation (8). For $B_* = \text{const}$ in the interval $[\tau_1, \tau_2]$, where $\tau_2 > \tau_1$

$$m(\tau, \tau_0) =$$

$$= \frac{E_1(\tau)-q_2E_1(\tau_0-\tau)+B_*[E_1|\tau-\tau_2|-E_1|\tau-\tau_1|]}{E_3(\tau)-q_2E_3(\tau_0-\tau)+B_*[E_3|\tau-\tau_2|-E_3|\tau-\tau_1|]}.$$

A multiplicity of other cases is examined analogously.

As we can see, for the exact determination of the temperature field we must determine the function $m(\tau, \tau_0)$ by the method, for example, of successive approximation; the boundary conditions can be determined exactly at the same time, as done, for example, in [20, 21]. However, this procedure is more complex than the direct solution of the integral equation. Therefore, it is recommended that we limit ourselves to an approximate study of the function $m(\tau, \tau_0)$ to provide a justification for diffusion approximation and for the selection of the number m with selection of the boundary conditions from [6, 18, 19].

NOTATION

q is the radiant flow vector, in W/m^2; m is the dimensionless function (number) under investigation; $k = \alpha + \beta$ is the attenuation factor, in m^{-1}; α and β are the absorption coefficients (reradiation) and dissipation; U is the radiant energy density, in J/m^3; P is the tensor of ray density (second rank), in J/m^3; P_{11} is its component determined by (4); I is the radiant flow intensity, in $W/m^2 \cdot$ ster; I_+ and I_- are the intensities in the positive and negative directions of the coordinate axis, respectively; $d\omega$ is the element of solid angle; θ is the angle between the axis of coordinates and a ray; T is the absolute temperature; l is the coordinate with its origin near surface 1, in m; l_0 is the layer thickness, in m; B_+ and B_- are identical to B_{ef} for both sides from the point adjacent to coordinate τ; q_{ef1} and q_{ef2} are the densities of effective hemispherical flows at the boundaries of the layer, in W/m^2; b_k is the coefficient of the odd power function $f_2(u)$, in W/m^2; a_* and b_* are the dimensionless coefficients explained in the text; g_0 is the specific power of sources independent of radiation, so that $g_0 = \text{div}\,\mathbf{q}$, W/m^3; $c = 3 \cdot 10^8$ m/sec; $\sigma = 5.68 \cdot 10^{-8}\ W/m^2 \cdot deg^4$; $\mu \equiv |\cos\theta|$; $u = |\tau' - \tau|$; $\tau = \int_l k\,dl$; $\tau_0 = \int_{l_0} k\,dl$; $q_2 = q_{ef2}/q_{ef1}$; $B_* = \pi B_{ef}/q_{ef1}$; $\pi B_{ef} = \sigma T^4 - (\beta/4\alpha k)g_0$; $E_n(\tau) = \int_1^\infty t^{-n}\exp(-\tau t)dt$.

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Ural Electromechanical Institute of Railroad Engineers, Sverdlovsk